How to Practice It?

An integrated approach to algebraic skills

Alex Friedlander and Abraham Arcavi
Weizmann Institute of Science - Israel

Recently, an increasing number of mathematics educators consider that procedural/transformational and conceptual/sense making activities are intrinsically complementary and should be thoroughly integrated (Kieran, 2004; Star, 2005, 2007).

What kinds of tasks, exercises and problems,would engage student learning in such a way that such complementarity is enacted? We propose to answer this question from the perspective of instructional design. The design we propose offers teachers and students opportunities for an effective teaching and learning of algebraic procedures. The tasks designed bear some similarities with those in traditional practices – short exercises and problems, accessible to and implementable in regular classrooms by regular teachers. On the other hand, these tasks require the adoption of some less widespread practices – such as the pursuit of alternative solutions, the evaluation of the effectiveness of approaches, class discussions, reflection on procedures and solution methods, and more. In the following, we present a framework we developed in order to guide the design of such tasks for beginning algebra learners (Resnick et al., 2007). The rationale and the structure of the framework can be summarized in a two-dimensional matrix (Table 1). One of the dimensions refers to the algebraic procedures pertaining to the different topics. The other dimension refers to the required/desirable cognitive processes as identified by research (e.g., Mason et al., 2005; Stacey et al., 2004; Bednarz et al., 1996; Sutherland et al., 2001) and by our own classroom and in-service teaching experience.

This framework is not exhaustive on either of its dimensions and the designed collection of tasks does not include elaborated inquiry problems which are longer in scope and richer in the required skills and processes.

Table 1. Framework for designing beginning algebra exercise sets.

Topic \ Cognitive process	Direct application of procedures	Reversed thinking	Global comprehension	Constructing examples or counter-examples	Identifying errors or misconceptions	Considering and justifying multiple choice tasks	Understanding the meaning of algebraic operations	Informal thinking	Divergent thinking
Negative numbers									
Algebraic expressions									
Equations, and inequalities									
Systems of equations									
Factorization									

Since the cognitive processes may be interrelated, the framework described here constitutes more of a heuristic guide for task selection or design than a clear-cut categorization. The development of the framework is based on the following premises:

- Algebraic skills are an important component of algebra instruction for *all* students.
- All students are capable to engage in the cognitive processes described above, independently of their mathematical ability.
- Effectiveness in the performance of algebraic skills constitutes an integral part of proficiency and learning.

- Short exercises are important since they support transparency of goals, visibility of control mechanisms and can provide a sense of short term accomplishments without requiring long term investments.
- Practice of algebraic skills should include “meta-issues” and the reflection upon them, creating opportunities for classroom discussions which go beyond the “correct/incorrect” as the only possible feedback.
- Raising the level of the cognitive demand encourages a variety of thinking processes, strategies and reflective practices.
- Even within learning procedures, creativity can and should be encouraged (e.g. by requiring students to design exercises of their own).

In the following, we describe and exemplify the nine processes chosen as components of the cognitive dimension in algebraic practice

1. ***Direct application of procedures*.** The principles and the tasks that correspond to this item are the most common type of exercises found in any textbook with a traditional approach.

2. ***Reverse thinking***. Students are required to reconstruct expressions or equations on the basis of given parts and/or the final result of an exercise. These tasks call for backward thinking, or for the reconstruction of a procedure already performed but missing. For example:

> Fill in the missing operation signs.
>
> a) 6m ○ 7 ○ 2m = 8m + 7
>
> b) 6m ○ 7 ○ 2m = 20m
>
> c) 6m ○ 7 ○ 2m = 44m

3. ***Global comprehension***. This ability involves dealing with multiple-term expressions as a single unit (Jensen & Wagner, 1981), rather than viewing it as a collection of “atomic” components (variables, numbers and operations). This approach requires a global "gestalt" view and the identification of "units" of reference in complex expressions. For example:

Knowing that $2x + 15 = -2$, find the values of the following expressions.

a) $2x + 16 =$ ______

b) $2x + 20 =$ ______

c) $2x + \ \ 5 =$ ______

d) $3 \cdot (2x + 15) =$ ______

e) $-1 \cdot (2x + 15) =$ ______

f) $-0.5 \cdot (2x + 5) =$ ______

4. ***Constructing examples or counter-examples***. In these tasks, performance involves understanding the meaning of a concept or a procedure, reversed thinking, justifying, and creative thinking. It also positions students in a "teaching" position and thus it may induce reflection about one's own or other's potential sources of difficulties. For example:

Complete the missing distracters in the quiz on equivalent expressions:

i. For each expression, write four choices – some correct and some incorrect.

ii. Ask a friend to solve your quiz.

Quiz

Mark √ by each equivalent expression.

Some expressions may have more than one equivalent.

1. $5 - x - 3 =$

a) b) c) d)

2. $7 - 2 \cdot (x - 3) =$

a b) c) d)

3. $4(x \cdot 5) =$

a) b) c) d)

4. $6 - (2 - x) =$

a) b) c) d)

5. ***Identifying errors or misconceptions*****.** Students are required to inspect and understand a solution produced by a real or a "designer-made" fictitious peer, identifying mistakes and their sources**.** For example:

Is the following solution correct? If not, make the necessary corrections.

$$2(3 - x) + 3(x - 2) + 7x = 2$$
$$6 - 2x + 3x - 6 + 7x = 2$$
$$8x = 2$$
$$x = 4$$

6. ***Considering and justifying multiple choice tasks.*** In these tasks, the solution process involves identifying a correct solution among a collection of incorrect options, considering multiple solution methods or answers, in-depth understanding of a concept, and critical thinking. For example:

Mark all the expressions that are equivalent to the boxed ones.

a) $\boxed{\frac{a-6}{3} =}$

$-\frac{a+6}{3}$ $\quad\frac{1}{3}a-6$ $\quad\frac{a}{3}-2$ $\quad\frac{6-a}{3}$

$\frac{1}{3}(a-6)$ $\quad a-6:3$

b) $\boxed{\frac{ab}{4} =}$

$\frac{a}{4}\cdot b$ $\quad\frac{a}{2}\cdot\frac{b}{2}$ $\quad\frac{1}{4}ab$ $\quad\frac{a}{4}\cdot\frac{b}{4}$

$\frac{a}{4}+\frac{b}{4}$ $\quad\frac{a+b}{4}$

c) $\boxed{\frac{x\cdot y}{3} =}$

$\frac{x}{3}\cdot y$ $\quad x\cdot\frac{y}{3}$ $\quad x\cdot y:3$ $\quad\frac{1}{3}(xy)$

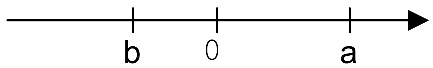
$\frac{x}{3}\cdot\frac{y}{3}$ $\quad(x\cdot y):3$

7. ***Understanding the meaning of algebraic operations.*** Students are required to work within and across multiple representations, finding

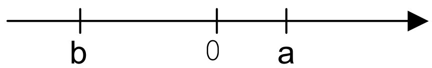
alternative solution methods to discuss and to reflect upon them. For example:

a and *b* represent numbers on the number line.

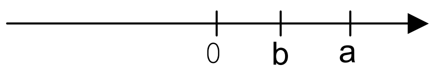
Mark the operations that produce a negative result for a $\square$ b.

a) 

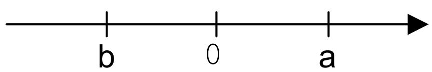
+ - x :

b) 

+ - x :

c) 

+ - x :

d) 

+ - x :

8. ***Informal thinking***. This ability involves predicting, monitoring, and interpreting resorting to common sense, conceptual understanding, and application of properties without explicitly using formal symbolic treatments. For example:

The average of four numbers is negative.

- Can it be that all four numbers are negative? If so, give an example.
- Can it be that all four numbers are positive? If so, give an example.
- Can it be that only two out of the four numbers are positive? If so, give an example.
- Can it be that only three out of the four numbers are negative? If so, give an example.

9. ***Divergent thinking.*** Students are required to produce or discuss multiple solution methods or answers to an exercise, giving a wide variety of examples or making unexpected connections with other domains. Students can also be invited to create and design. For example:

Complete the following to obtain identities in different ways.

____)^2 + (___ = ___ + 6x + ___ a)

____)^2 + (___ = ___ + 6x + ___ b)

____)^2 + (___ = ___ + 6x + ___ c)

____)^2 + (___ = ___ + 6x + ___ g)

Initial observations of student work in several experimental classes on samples of exercise sets of these kinds indicate that the work promoted students' meaningful learning, improved their procedural knowledge, increased motivation and even provided some opportunities for the development of symbol sense (as defined by Arcavi, 1994, 2005). Most observed low achievers were able to complete a variety of exercises of lower difficulty level but similar in spirit.

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