Designing Questions to Probe Relational or Structural Thinking in Arithmetic

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There is a strong case for arguing that the application of relational thinking to solve number sentences embodies features of mathematical thinking that are centrally important. This paper looks at the design and implementation of a questionnaire used in three countries to examine students' capacity to use relational thinking to solve different types of number sentences. Some students appeared to rely solely on computational method to solve number sentences with one missing number whereas other students were able to demonstrate clear use of relational strategies. Did those who used computational approaches do so out of preference, because they were good at computation, or did they do so because computation was the only strategy they could use? The questionnaire needed to discriminate between these two groups and to design specific questions that required students to think relationally.

A Design Issue

How do we probe more deeply into connections between structural thinking in arithmetic, on the one hand, and mathematical structure, on the other, to learn more about shifts from particular to structural understandings? The importance of structural understandings in these contexts is that they offer students a source of control which allows them to move beyond the particular situation. In designing a research instrument we need to capture the extent to which this control is open to growth – that is, it is open to increasing levels of generality.

Perceptions of Structure and Number Sentences

Carpenter and Franke (2001), Stephens (2006), Jacobs, Franke, Carpenter, Levi and Battey (2007), Molina (2007), and Fujii and Stephens (2001, 2008) have studied in detail ways in which children as young as 6-years-old respond when asked to justify their decision about the validity of statements such as 173 – 35 + 35 = 173.

Some children calculate their way to the answer and then decide; some start to calculate and then notice the familiar number to be subtracted and declare their decision; others look at the expression and declare immediately without apparently doing any calculation at all. To decide without any calculation is a form of relational thinking, of appreciating arithmetic structure concerning, if not zero, then the effect of first subtracting and then adding the same quantity. It could be the manifestation of a fundamental awareness that taking and then replacing makes no change (Lakoff & Nunez, 2000). As such it would be an example of a theorem-in-action (Vergnaud, 1983).

Several authors, including Carpenter and Franke (2001) and Stephens (2006), refer to the thinking underpinning this kind of thinking as *relational thinking,* but it might just as easily be called *structural* thinking. Structural thinking is in this sense productive – pointing to the fact that the products of structural thinking can extend from being able to give several other instances of the same property to giving fully developed generalisations.

However students' appreciation of the relation implicit in 173 – 35 + 35 = 173 may go beyond its use in this context; they may be aware that they are using a generality – indeed, they may be able to state that they are using an abstract relation. When learners are justifying their decisions to each other, it is often very difficult to decide whether they are aware of a general property (the 173 and the 35 are instances of generality) or whether the 173 is mentally fixed but the 35 is an instantiation of taking and replacing, or an instantiation of $173 - a + a = 173$, or even of $b - a + a = b$, that is, whether the 173 and the 35 are seen as mere place holders or as quasi-variables (Fujii & Stephens, 2001; Lins & Kaput, 2004; Fujii & Stephens, 2008) or as mentally fixed for the time being.

Some children can enact one or other of these relationships without being able to bring it to explicit articulation, and may not even use it robustly in all instances. Young children can sometimes articulate the general structural principles underlying the relationship, for example as "if you start with a number and you take away something and give it back you haven't changed the starting number". Children may be fuzzily aware that this relationship holds for all a and b with which they are familiar, or even able to express it as a generality, yet they may not have encountered or considered situations where $a > b$ or where a and b are negative or rational. Thus general statements may adequately express limited structural understanding, based on restricted ranges of change.

One way to think about the different possibilities, and even to seek evidence for different awarenesses, is through the focus and structure of their attention. The way they describe what they are doing sometimes suggests not only what they are attending to, but different ways in which they are attending, whether to the particular, or through the particular to the general, or at the particular though the general. Another way of expressing the complexity of learner awareness is that, without further probes, it is difficult to know the range-of-permissible-change of which the learner is aware, and even which dimensions-of-possible-variation the learner is contemplating. By asking learners to construct similar examples, some light is shed on at least some of the features they appreciate as changeable as well as the range over which the learner accepts that the change can be made (Watson & Mason, 2005).

Structural awareness, or relational thinking in this context, therefore, involves explicit awareness of some range-of-permissible-change of some dimensions-of-possible-variation. These ranges-of-permissible-change can be extended when other kinds of numbers and number-like objects are encountered.

Designing and using Missing Number Sentences

Research carried out by Stephens (2008), Stephens and Wang (2008), and Stephens, Wang and Al-Murani (2008) used three types of mathematical sentences to explore students' capacity to think about important aspects of mathematical structure. Type I number sentences used one missing number, Type II number sentences used two missing numbers and Type III sentences were modelled on Type II using algebraic symbols. In the three studies reported above, these three types of questions were used with 275 students in Australia, China and England ranging from Year 6 (10-11 years old) to Year 9 (13-14 years old). Since then the same questions have been used successfully with children of similar ages – from upper primary and junior secondary classes – in Indonesia, Brazil and Japan.

Type I: one missing number

The first kind of number sentence (Type I) presents students with a number sentence with one number missing and asks them to find the value of the missing number and to explain briefly the reasoning they used to reach a solution. The authors named above used all four operations in Type I, and invited students to find the value of a missing number and to explain their thinking. For each operation, four different problems similar to those above were used but with the unknown number being set in a different place for each of the four problems. The suite of Type I questions used for addition is shown in Appendix 1. The following four Type I questions illustrate each of the four operations:

$$c + 17 = 15 + 24$$
$$99 - c = 90 - 59$$
$$48 \times 2.5 = c \times 10$$
$$3 \div 4 = 15 \div c$$

Some students relied on computation to solve these problems. In each case, they first computed the result of the operation involving the two known numbers, and then used this result to calculate the value of the missing number on the other side of the equal sign. Other students used compensation and equivalence. Irwin and Britt (2005) claim that the methods of compensating and equivalence that some students use in solving these number sentences may provide evidence of "what could usefully be described as structural thinking" (p. 169). They give, as an example, the expression 47 + 25 which can be transformed into an equivalent expression 50 + 22 by adding 3 to 47 and subtracting 3 from 25, thus making calculation easier. They also claim that "when students apply this strategy to sensibly solve different numerical problems they disclose an understanding of the relationships of the numbers involved. They show, without recourse to literal symbols, that the strategy is generalisable" (Irwin & Britt, 2005, p. 171).

It is however not always so easy to deduce from observed behaviour whether learners are aware of the 47 and 25 as dimensions-of-possible variation, of 3 as a dimension-of-possible-variation, or of the adding and subtracting as a special instance of more general compensation (another dimension-of-possible-variation). Structural thinking is much more than seeing a pattern, such as 'when one number increases by three the other goes down by three'.

Where this merely recounts the pattern used in this particular problem with no sense of generalisation to other instances, it indicates recognition of a relationship in particular but not perception of property in general. A capacity to generate other instances that illustrate the same property is a feature of structural thinking. Often it seems that students act as if they have some such awareness, but it may be neither robust nor universal. Furthermore, their perceived range-of-permissible change may be confined to positive whole numbers rather than to numbers more generally, whether involving negatives, rationals or decimals. A great deal depends on whether they are attending to and dwelling in the particular or in some sense aware of a property being instantiated, whether that in-dwelling comes from an awareness as a basis for their action in the form of a theorem-in-action, or from an emerging behavioural practice.

Where learners respond to direct suggestions to 'use compensation' or to 'add and subtract', or to indirect prompts to use a strategy before trying to do it directly, they are on the way to being influenced by careful scaffolding and fading (Seeley Brown, Collins & Duguid, 1989; Love & Mason, 1992) so as to be able to initiate these actions for themselves (van der Veer & Valsiner, 1991). Some where along the line, they display structural awareness. A deep understanding of equivalence and compensation is at the heart of structural thinking in arithmetic. Students need to know the direction in which compensation has to be carried out in order to maintain equivalence (Kieran, 1981; Irwin & Britt, 2005). Indeed, it may be argued that structural thinking is present only when students' explanations show that they understand the fundamental importance of the operations involved, make use of equivalence, the direction of compensation required to maintain equivalence, and how particular results are part of a more general pattern.

According to Stephens (2008), students used a range of equally successful explanatory methods in their written responses to Type I sentences. Some students used arrows or brackets or other notation in ways which indicate a comprehensive understanding of equivalence and compensation. The use of arrows or directed lines to connecting related numbers, such as from 2.5 to 10, showing ×4 above the line or arrow, and an arrow connecting 48 to the unknown number, with ÷4 joined to this line or arrow, was a simple and effective method of demonstrating the direction of compensation. Other students wrote their thinking in the form of mini-arguments (see Vergnaud, 1983) using expressions such as "Since 17 is two more than 15, the missing number has to be two less than 24 in order to keep the balance". Other students chose to make a similar argument starting with a relationship between 17 and 24. Relational thinking can be expressed using a wide range of methods and forms, but in all cases these forms and methods draw attention to the fundamental ideas of equivalence, and compensation as required by the particular operations. These features are equally important to elucidating the structures of the three types of mathematical sentences we refer to.

One of the difficulties encountered in relying on Type I number sentences in a written questionnaire is that some students who may be quite capable of using structural thinking nevertheless choose to solve Type I sentences by computation. While they may find computation attractive and easy, these students need to be distinguished from those who are *restricted* to solving such sentences computationally. This important distinction can, of course, be explored by means of an interview; by asking, for example, "Could you have solved this number sentence in another way?".

Type II: two missing numbers

But there are other ways of pushing students beyond computation using written responses. In studies referred to above, this was achieved through the use of Type II number sentences using two unknowns, denoted by Box A and Box B, and employing one arithmetical operation at a time. Type II questions are exemplified in parts (a) to (d) in Figure 1 below. Using a similar template, other questions were devised involving subtraction, multiplication and division. By asking students to construct similar examples, some light is shed on at least some of the features they appreciate as changeable and how these features are specified mathematically.

Almost all students were able to make up three replicas of each mathematical sentence using specific numbers. In dealing with addition, some students used large numbers such as 1,000,000 in Box A and 999,998 in Box B; and others used decimal numbers and fractions. There were students who chose quite simple numbers such as 3, 4, and 5 in Box A which they associated with 1, 2, and 3 respectively in Box B. Those who used more complex numbers in Box A and Box B usually had no difficulty in describing in part (b) the relationship between the numbers in Box A and Box B and in successfully answering the subsequent questions. But the same was true for many who had used relatively simple numbers in their exemplifications of the mathematical sentence in part (a).

Think about the following mathematical sentence:

$$18 + c = 20 + c$$

Box A **Box B**

(a) Can you put numbers in Box A and Box B to make three correct sentences like the one above?

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

Figure 1. Type II number sentence involving addition.

What actually discriminated between students' accomplished and not-so-successful responses to parts (c) and (d) was how they answered part (b). Almost all students were able to identify some *pattern* between the numbers in Box A and the numbers they had used in Box B. But simply seeing *a pattern* may not be productive in perceiving *structure* as a property to be instantiated elsewhere.

Some students identified what might be called a non-directed relation between the numbers used in Box A and Box B, saying, for example, "There is 2 difference", or "They are 2 apart", or "There is a distance of 2". Some qualified this non-directed relation by saying, "There is always 2 difference". Others noticed a directed relation between the numbers used but attached no magnitude to the relation, saying, for example, "Box A is bigger than Box B". Others expressed a direction but without referring to Box A or Box B, saying, for example, "One number is always higher than the other number by 2", or "One is two more than the other". These responses illustrate clearly the difference between seeing only particular features of a relationship and what we would call structural or fully referenced relational thinking.

In each of these cases, students had noticed some relationship between the numbers in Box A and Box B, but their descriptions suggest that they were attending to a specific feature of the relationship that could be expressed comprehensively as "The number in Box A *is* two more than the number in Box B". On the other hand, it may be that when they came to articulate what they were aware of, their attention was diverted to a part rather than some more comprehensive whole. Many might not have been familiar with the kind of relationships which prove to be productive in mathematics. To be productive, relationships have to be fully referenced − in this instance, there has to be unambiguous reference to the numbers represented by Box A and Box B; and the magnitude and direction of the relationship has to be specified − just saying that one is bigger than the other, or that there is a difference of two is not enough.

Students had their own ways of elaborating comprehensive descriptions; with some using logical qualifiers such as "must be" or "has to be" instead of "is", whereas others added a phrase like "in order for the sentence to be correct" or "for both sides to be equivalent". There were others who chose to write the relationship in symbolic form, writing an equation involving Box A and Box B, or in some cases just A and B. The presence of logical qualifiers and the use of symbolic forms is evidence that students had grasped a source of control that comes from fully referenced relational thinking. These students appeared to be attending more carefully to what we recognised as the structure of the mathematical sentence than those above whose statements pointed to some but not all of the features essential for equivalence.

What we find very illuminating in all responses to the questionnaire in the studies reported above is that no student who referred to only partial features of the relationship between the numbers in Box A and the numbers in Box B answered part (d) successfully. Of course, many attempted to answer this question but their answers were always incomplete. Some students answered "No", but then added that it would be necessary to have numbers in Box B that "will allow both sides to balance". Others thought that it would be impossible without using negative numbers. Still others continued to rely on the partial features that they had used in answering part (b) in order to answer part (d).

In summary, these kinds of responses may not be so much incorrect and erroneous as incomplete. They fall short in various ways of being productive. They add weight to the distinction we want to underline that seeing some relationship or pattern is not the same as recognising a mathematical structure. We do need to point out that a mathematically complete description of the relationship between the numbers in Box A and the numbers in Box B, as required for part (b), did not guarantee a successful answer to part (d). Some who had correctly answered part (b) appeared to be worried about the range of variation that might be required for part (d) to be correct. Nevertheless, there was a strong association between a correct response to part (b) and part (d). Furthermore, when students used similar partial or incomplete descriptions to describe the relationship between the numbers in Box A and the numbers in Box B in related questions involving subtraction, multiplication and division, they were also unable to successfully answer the corresponding part (d) question, "If you put any number in Box A, can you still make a correct sentence?"

Type III: symbolic sentences

Following part (d) students were given a sentence involving literal symbols c and d in place of the boxes and where the numbers were slightly different. In the case of addition (Figure 1) a symbolic relationship of the form $c + 2 = d + 10$ was used in a part (e). By asking learners to deal with a sentences that is symbolically related to its corresponding Type II sentence, some clearer light is shed on the range over which the learner accepts that the change can be made. The single-page format in which Type II and Type III sentences relating to addition are also shown given in Appendix 1.

Students were asked, "What can you say about c and d in this mathematical sentence?" Once again, none of the students who had given one of the 'partial descriptions' of the full relationship between the numbers in Box A and the numbers in Box B successfully answered this question. Many chose to give a particular set of values, such as $c = 10$ and $d = 2$ in the case of addition. Those who were able to answer part (e) successfully had all given a complete and correct response to part (b) and part (c), and most had given a correct and complete response to part (d). Among successful responses, there was, moreover, a high level of consistency between the language and terminology used to explain students' answers to part, (d) and (e). For example, where students had answered (d) using a symbolic relationship they almost always used a symbolic expression to describe the relationship between *c* and *d*; and where they had used written descriptions in answering part (d) and part (e) they used similar words and phrases in both expressions. One student commented that the *c* and *d* were "just like Box A and Box B". This suggests an aspect of structural understanding that could be explored more deeply in interviewing students who gave correct answers to parts (d) and (e). Referring to the Type II number sentence and its corresponding Type III symbolic expression involving *c* and *d,* students could be invited to comment on the statement: "These two sentences look different. Are they so different? Can you comment from a mathematical point of view on any similarities you notice about them?"

How well did the design work?

The use of Type I number sentences did give many students an opportunity to demonstrate their capacity to use relational thinking. Those who did confidently on Type I sentences continued to exhibit relational thinking, usually very competently, on subsequent Type II and type III sentences. There were other students who used a mix of computational and relational approaches on Type I sentences. When these students were required to use relational methods on Type II and Type III sentences, the extent of their competence in relational thinking could be more clearly assessed. Some of these students appeared to be students who found computing the result of a Type I sentences easy to do, but who could have just as easily have used relational thinking, but chose not to, or simply thought that computation was the easiest way to go.

Those most strongly affected and clearly identified by the design of the questionnaire were those who consistently used computational approaches on Type I. In China, for example, there were more students than in the other countries who solved Type I sentences by computation and yet were quite capable of working relationally on Type II and Type III questions. The use of Type II questions clearly identified those who had used computation on Type I sentence by choice – the choosers – in contrast to those who were reliant on computation to solve Type I sentences and were not able to access relational methods.

The first group was able to "shift" as required into relational thinking and could demonstrate various levels of competence on Type II and Type III sentences. The second group was able attempt Type II questions over the four operations, but typically failed to identify fully the mathematical relationships between the numbers in Box A and Box B. They certainly found Type III sentences involving symbolic terms difficult to answer except in terms of giving a specific pair of values for *c* and *d*.

There were opportunities to interview several students who had shown clear relational thinking on Type II questions but who had answered Type I questions typically by computation. In interview, these students were first given an opportunity to review shown their successful handling of Type II questions for one of the operations. Then they were asked to look at their solutions to the corresponding operation where they had used only computation. The interviewer started by saying: "These are answers you have calculated correctly, but is there another way you might have thought about these questions?"

In most cases, it came as a surprise to these students that they could actually solve Type I questions relationally. Some saw these possibilities intuitively having considered their solutions to Type II questions. However, others had to "coach themselves" to make Type I questions be seen as structurally similar to Type II questions. For example, when looking at c + 17 = 15 + 24, one student covered up c and 15 with his fingers leaving only 17 and 24 either side of the equal sign. And then said, "There is a 7 more on this side (pointing to the 24) and then lifted his fingers to direct attention to c and 15, then arguing that the number represented by c had to be seven more then 15. Likewise, these students could look at the 23 and the 26 in first Type I addition problem and see that a relationship of "three more" has to be compensated by "taking three off" 15 to give a value of 12 for the missing number. This pattern of covering up the number represented by the box in a Type I question and its corresponding number placed on the other side of the equal sign was a powerful way of identifying how the other two given numbers were related according to the operation used.

Learning to attend to a given pair of numbers in terms of the operation used, blocking out the other two numbers including the missing number shown by c helped some students to see a relationship that was not evident to them before. By working through several examples in this way, one can work with particular relationships. Type II and Type III questions were needed to establish if students are able to generalise these relationships. Other students, regrettably, found applying relational strategies to Type I sentences quite challenging. These would appear to need more explicit teaching, but this interview procedure did show that seeing a relationship *not-seen-before* was achieved by deliberately changing the focus and structure of attention of students who were given a suitable prompt from their own successful relational strategies on Type II sentences. This interview strategy was not extended to helping students deal with Type III sentences where many had difficulty giving a generalised answer, but it could be tried.

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Appendix 1

NAME YEAR LEVEL

Number Sentences and Relationships Questionnaire

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$23 + 15 = 26 + \square$

$73 + 49 = \square + 47$

$43 + \square = 48 + 76$

$\square + 17 = 15 + 24$

2. Can you think about the following mathematical sentence:

$$18 + \underset{\text{Box A}}{\boxed{}} = 20 + \underset{\text{Box B}}{\boxed{}}$$

(a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$$18 + \underset{\text{Box A}}{\boxed{}} = 20 + \underset{\text{Box B}}{\boxed{}}$$

$$18 + \underset{\text{Box A}}{\boxed{}} = 20 + \underset{\text{Box B}}{\boxed{}}$$

$$18 + \underset{\text{Box A}}{\boxed{}} = 20 + \underset{\text{Box B}}{\boxed{}}$$

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

(e) What can you say about c and d in this mathematical sentence?

$$c + 2 = d + 10$$